[MAT CC 06]

Part II

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Power Series

2014-2016, 2011

A series of the form

$$\sum_{n=0}^{\infty} a_n z^n \text{ or } \sum_{n=0}^{\infty} a_n (z-a)^n$$

is called a power series, where a_n and a are complex constants and z is a complex ~~number~~ variable.

If $z - a = \xi$ (complex variable)

Then: $\sum_{n=0}^{\infty} a_n (z-a)^n = \sum_{n=0}^{\infty} a_n \xi^n$ which is in the first form.

so, first and second both are in the same form.

Absolute convergence of $\sum a_n z^n$:

The power series $\sum a_n z^n$ is said to be absolutely convergent if the series $\sum |a_n| |z|^n$ is convergent.

The power series $\sum a_n z^n$ is said to be conditionally convergent if $\sum a_n z^n$ is cvg. but $\sum |a_n| |z|^n$ is not cvg.

Radius of convergence of power series: 2011, 2014-16

let $\sum a_n z^n = \sum u_n$ (absolutely)

The series $\sum u_n$ is convergent if $\lim_{n\to\infty} |u_n|^{1/n} < 1$ (Cauchy Root test)

and divergent if $\lim_{n\to\infty} |u_n|^{1/n} > 1$ (Cauchy Root test)

~~This implies~~ If $\lim_{n\to\infty} |a_n z^n|^{1/n} < 1$

then ~~it~~ $\lim_{n\to\infty} |a_n|^{1/n} |z| < 1$

Taking $\lim_{n\to\infty} |a_n|^{1/n} = \frac{1}{R}$

then we get: $\frac{|z|}{R} < 1$ i.e. $|z| < R$

Thus $\sum a_n z^n$ is cvg. when $|z| < R$

and similarly div. when $|z| > R$.

so, corresponding to every power series, there exists a positive (non-negative) real number R such that, if the series is cvg then $|z| < R$

and if the series is div. then $|z| > R$.

Now we consider a circle with centre at the origin and of radius R. Then:

(i) The power series is cvg. for every z within the circle.

and (ii) the power series is div. for every z outside the circle.

Such a circle is called circle of ~~curvature~~ convergence and its radius is called radius of ~~curvature~~ convergence of the power series $\sum a_n z^n$.

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Now three cases will arise:

(i) when $R=0$

In this case the series is cvg. only when $z=0$.

(ii) when R is finite.

In this case the series is cvg. at every point within the circle and div. at every point outside the circle.

(iii) when R is infinite.

In this case the series is cvg. for every point z.

2014, 2012, 2010 Cauchy Hadamard Theorem:

For every power series $\sum_{n=0}^{\infty} a_n z^n$, there exists a number R such that $0 \le R \le \infty$ with the following properties:

(i) The series converges absolutely for every z such that $|z|<R$

(ii) The series diverges if $|z|>R$

Proof: write the previous topic

i.e. Radius of curvature of power series.

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Abel's Theorem. 2013, 2014-16

Q3. If the power series $\Sigma a_n z^n$ converges for a particular value z_0 of z, then it converges absolutely for all values of z for which $|z| < |z_0|$

Proof: Let the series $\Sigma a_n \cdot z_0^n$ converges. Then its nth term $a_n \cdot z_0^n$ must tend to 0 when $n \to \infty$

i.e. $\lim_{n\to\infty} a_n \cdot z_0^n = 0$.

So, there exists a positive number M such that

$|a_n \cdot z_0^n| \le M$, for all n.

i.e. $|a_n \cdot z^n| \le M \left|\frac{z}{z_0}\right|^n$

It is given that $|z| < |z_0|$, so $\left|\frac{z}{z_0}\right| < 1$ and therefore the geometric series $\sum_{n=0}^{\infty} \left|\frac{z}{z_0}\right|^n$ is convergent

Hence by comparison test the series $\Sigma |a_n z^n|$ is convergent for all z for which $|z| < |z_0|$

Therefore the power series $\Sigma a_n \cdot z^n$ is absolutely convergent for all z satisfying $|z| < |z_0|$

Proved.

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Problems.

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1. Find the radius of ~~curvature~~ convergence of the following power series:

2014, 2009 (i) $\sum \frac{z^n}{n^n}$ Q8. (ii) $\sum \frac{2^{-n} \cdot z^n}{1+in^2}$ (iii) $\sum \frac{(\lfloor n)^2}{\lfloor 2n} \cdot z^n$

Q8 (iv) $\sum \frac{\lfloor n}{n^n} \cdot z^n$ 2014, 2012, 2010

Sol^n: (i) The given series is $\sum a_n \cdot z^n = \sum \frac{z^n}{n^n}$

so: $a_n = \frac{1}{n^n}$

$\therefore \frac{1}{R} = \lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \left(\frac{1}{n^n}\right)^{1/n} = \lim_{n\to\infty} \frac{1}{n} = \frac{1}{\infty} = 0.$

~~$\therefore R = \frac{1}{\infty} = 0.$~~

$\therefore \frac{1}{R} = 0$ so, $R = \frac{1}{0} = \infty.$

so radius of curvature $(R) = \infty.$

(ii) Here: $a_n = \frac{2^{-n}}{1+in^2}$

$\therefore \frac{1}{R} = \lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \left|\frac{2^{-n}}{1+in^2}\right|^{1/n} = \lim_{n\to\infty} \frac{1}{2(1+n^4)^{1/2n}}$

Since $|1+in^2| = \sqrt{1+n^4}$

$= \lim_{n\to\infty} \frac{1}{2}(1+n^4)^{-1/2n}$

~~$= \frac{1}{2} \lim_{n\to\infty} \left[1 - \frac{1}{2n} n^4 + \cdots\right]$~~

$= \lim_{n\to\infty} \frac{1}{2}\left[n^4\left(1+\frac{1}{n^4}\right)\right]^{-1/2n}$

$= \lim_{n\to\infty} \frac{1}{2} \cdot (n^4)^{-1/2n} \left(1+\frac{1}{n^4}\right)^{-1/2n}$

$= \lim_{n\to\infty} \frac{1}{2} \cdot \frac{1}{(n^4)^{1/2n}} \cdot \left(1+\frac{1}{n^4}\right)^{-1/2n}.$

$=$ ~~$\lim_{n\to\infty} \frac{1}{2(n^4)^{1/2n}}$~~ $= \lim_{n\to\infty} \frac{1}{2 \cdot n^{2/n}} \left(1+\frac{1}{n^4}\right)^{-1/2n}$

$= \lim_{n\to\infty} \frac{1}{2 \cdot (n^{1/n})^2}\left[1 - \frac{1}{2n^5} + \cdots\right]$

$= \frac{1}{2} \cdot 1 \cdot 1,$ Since, $\lim_{n\to\infty} (n^{1/n}) = 1.$

$= \frac{1}{2}$

So: $\frac{1}{R} = \frac{1}{2}$ $\therefore \boxed{R = 2}$ A

(iv) Here: $a_n = \frac{\lfloor n}{n^n}$

$\therefore a_{n+1} = \frac{\lfloor n+1}{(n+1)^{n+1}}$

$\therefore \frac{a_{n+1}}{a_n} = \frac{\lfloor n+1}{(n+1)^{n+1}} \times \frac{n^n}{\lfloor n}$

$= \frac{n+1}{(n+1)^{n+1}} \cdot n^n$

$= \left(\frac{n}{n+1}\right)^n$

$= \frac{1}{\left(1+\frac{1}{n}\right)^n}$

$\therefore \frac{1}{R} = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$

$= \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n}$

$= \frac{1}{e}$

$\therefore R = e.$

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(iii) $\sum a_n \cdot z^n = \sum \frac{(\lfloor n)^2}{\lfloor 2n} \cdot z^n$

so, $a_n = \frac{(\lfloor n)^2}{\lfloor 2n}$ $\therefore a_{n+1} = \frac{(\lfloor n+1)^2}{\lfloor 2(n+1)} = \frac{[\lfloor n\,(n+1)]^2}{\lfloor 2n+2}$

so: $a_{n+1} = \frac{[\lfloor n\,(n+1)]^2}{\lfloor 2n \cdot (2n+1)(2n+2)}$

$\therefore \frac{a_{n+1}}{a_n} = \frac{[\lfloor n\,(n+1)]^2}{\lfloor 2n\,(2n+1)(2n+2)} \times \frac{\lfloor 2n}{(\lfloor n)^2} = \frac{(n+1)^2}{(2n+1)(2n+2)}$

$= \frac{(n+1)^2}{(2n+1) \cdot 2(n+1)} = \frac{n+1}{2(2n+1)}$

$= \frac{1+\frac{1}{n}}{2 \cdot 2(1+\frac{1}{2n})}$

$\therefore \frac{1}{R} = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \frac{1+0}{4(1+0)} = \frac{1}{4}$

so, $R = 4$. Ans.

Q. Find the radius of convergence (~~curvature~~) of the power series:

2012, 2009 (i) $\sum_{n=0}^{\infty} (3+4i)^n \cdot z^n$ (ii) $\sum_{n=0}^{\infty} \frac{(-1)^n \cdot (z-2i)^n}{n}$

(iii) $\sum_{n=1}^{\infty} (\log n)^n \cdot z^n$ (iv) $\sum_{n=0}^{\infty} \left[\frac{n\sqrt{2}+i}{1+2in}\right] \cdot z^n$ 2010

and (v) $\sum_{n=0}^{\infty} \frac{1}{n^p} \cdot z^n$

Solⁿ: (i) Here: $a_n = (3+4i)^n$. $\therefore |a_n| = |(3+4i)^n| = (\sqrt{3^2+4^2})^n = 5^n$

$\therefore \frac{1}{R} = \lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} (5^n)^{1/n} = 5$

so, $R = \frac{1}{5}$ Ans.

(ii) we have: $\sum \frac{(-1)^n (z-2i)^n}{n}$

Comparing this series with $\sum a_n (z-a)^n$ we find $a = 2i$

$a = 2i$ which is the centre of circle of convergence.

and: $a_n = \frac{(-1)^n}{n}$ $\therefore a_{n+1} = \frac{(-1)^{n+1}}{n+1}$

$\therefore \frac{a_{n+1}}{a_n} = -\left(\frac{n}{n+1}\right)$ $\therefore \frac{1}{R} = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = 1$

i.e. $\frac{1}{R} = 1$

$\therefore R = 1$. Ans

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(iii) $a_n = (\log n)^n$ $\therefore \frac{1}{R} = \lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} [(\log n)^n]^{1/n}$

$$= \lim_{n\to\infty} \log n = \infty$$

So, $R = \frac{1}{\infty} = 0$. Ans

(iv) Here: $a_n = \frac{n\sqrt{2}+i}{1+2in}$ $\therefore |a_n| = \left[\frac{2n^2+1}{1+4n^2}\right]^{1/2} = \left[\frac{2\cancel{n^2}(1+\frac{1}{2n^2})}{4\cancel{n^2}(1+\frac{1}{4n^2})}\right]^{1/2}$

$\therefore |a_n| = \frac{1}{\sqrt{2}}$ $\therefore \frac{1}{R} = \lim_{n\to\infty} |a_n| = \frac{1}{2}$

$\therefore R = 2$ Ans

(v) Here: $a_n = \frac{1}{n^p}$ $\therefore a_{n+1} = \frac{1}{(n+1)^p}$

$\therefore \frac{a_{n+1}}{a_n} = \frac{n^p}{(n+1)^p} = \left(\frac{n}{n+1}\right)^p = \left(\frac{1}{1+\frac{1}{n}}\right)^p$.

$\therefore \frac{1}{R} = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left[\frac{1}{1+\frac{1}{n}}\right]^p = 1$

So, $R = 1$. Ans.

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Prob: Find the radius of convergence of the power series $f(z) = \sum_{n=0}^{\infty} \frac{z^n}{2^n+1}$ and prove that $(2-z)f(z) - 2 \to 0$ as $z \to 2$

~~200~~ [~~2013~~, 2011] 2014–2016

Solⁿ: We have: $f(z) = \sum_{n=0}^{\infty} \frac{z^n}{2^n+1}$

$$\therefore a_n = \frac{1}{2^n+1}, \quad a_{n+1} = \frac{1}{2^{n+1}+1}$$

$$\therefore R = \lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \lim_{n\to\infty} \frac{2^{n+1}+1}{2^n+1} = \lim_{n\to\infty} \frac{2^{n+1}(1+2^{-(n+1)})}{2^n(1+2^{-n})}$$

$$= \lim_{n\to\infty} \frac{2(1+2^{-(n+1)})}{1+2^{-n}}$$

$$= \frac{2(1+0)}{1+0} = 2.$$

So, Radius of convergence $= \underline{2}$

Now: $$f(z) = \sum_{0}^{\infty} \frac{z^n}{2^n+1} < \sum_{0}^{\infty} \frac{z^n}{2^n} = 1 + \frac{z}{2} + \left(\frac{z}{2}\right)^2 + \left(\frac{z}{2}\right)^3 + \cdots$$

$$= \left(1 - \frac{z}{2}\right)^{-1} = \left(\frac{2-z}{2}\right)^{-1} = \frac{2}{2-z}.$$

$$\therefore \lim_{z\to 2} (2-z) f(z) = \lim_{z\to 2} (2-z) \cdot \frac{2}{(2-z)} = 2$$

Hence: $(2-z) f(z) - 2 \to 0$, as $z \to 2$

<u>Proved.</u>

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Prob: Find the radius of convergence of the power series $\sum_{n=1}^{\infty} [2+(-1)^n] z^n$. $\frac{08}{[27]}$ [2013]

Sol: Here: $a_n = 2+(-1)^n$.

$\therefore |a_n|^{1/n} = [2+(-1)^n]^{1/n}$.

So, $\frac{1}{R} = \lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} |2+(-1)^n|^{1/n}$

$= \lim_{n\to\infty} \begin{cases} (2+1)^{1/n}, & \text{when } n \text{ is even.} \\ (2-1)^{1/n}, & \text{when } n \text{ is odd.} \end{cases}$

$= \lim_{n\to\infty} 3^{1/n} \text{ or } 1^{1/n}$

$= 3^0 \text{ or } 1^0 = 1.$

$\therefore \frac{1}{R} = 1$

$\Rightarrow R = 1$

So, radius of convergence $= \underline{1}$ Ans.

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